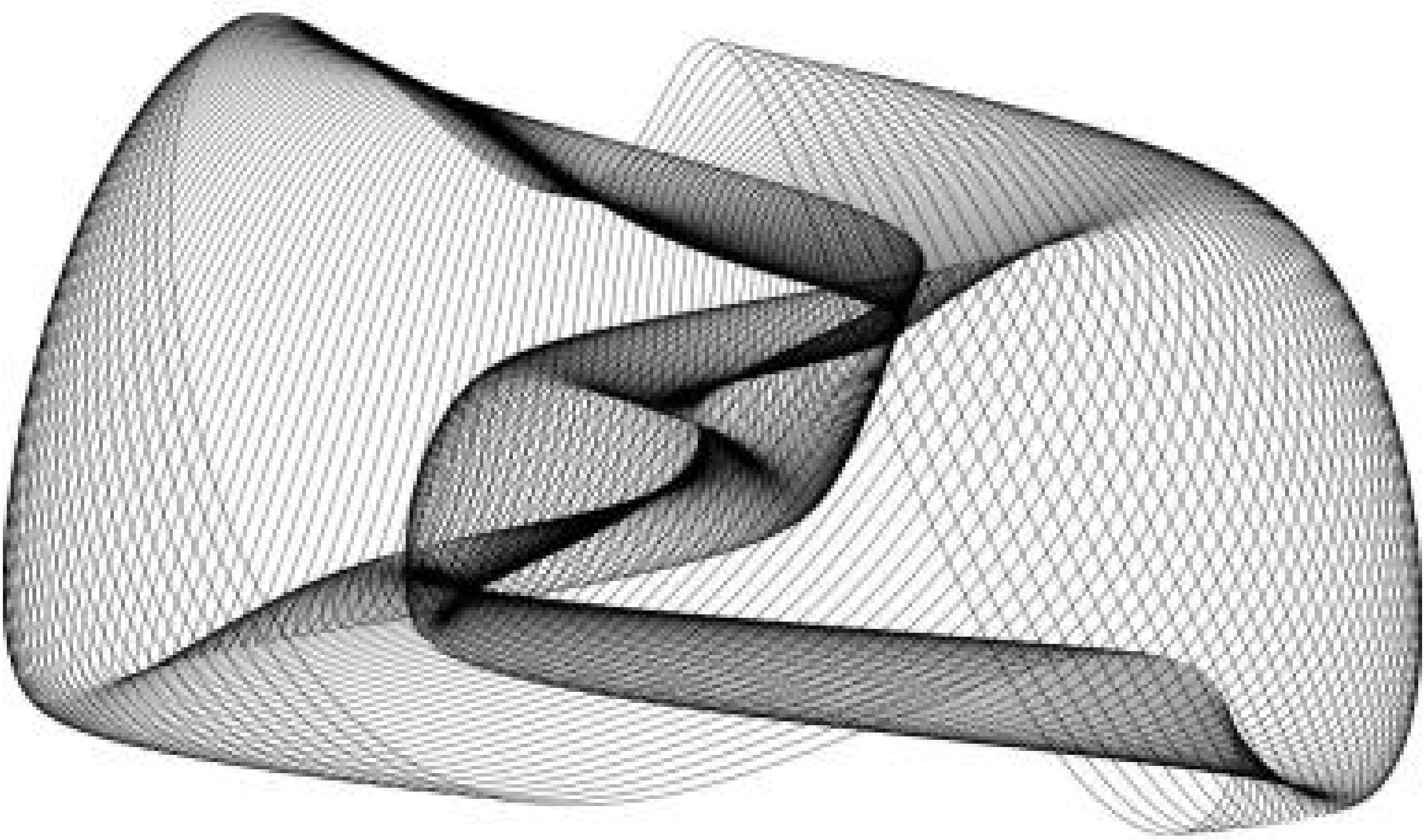
# MATHEMATICA

2020-22



The Mathematics Society
St. Stephen's College, Delhi

# Our Sincere Thanks To

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# CONTENTS

# ABOUT US

THE MATHEMATICS SOCIETY

"On a beautiful mind, there was a wall of Math"
-Josh Lucas

People usually have a preconceived notion that the Mathematics Society caters only to the geeks who are crazy about Mathematics, but that's not entirely true. Mathematics is an extremely versatile subject that can be found everywhere, ranging from Philosophy, Economics, and Sciences to Poetry.

The main aim of the Mathematics Society is to help you see the beauty of the subject and foster your creative mathematical ideas in its various forms. We are amongst the oldest and biggest departmental societies in college.

Throughout the year, our society conducts numerous enjoyable events, talks and workshops by eminent mathematicians, organizes discussions and releases publications, in an attempt to share the joys of pursuing mathematics with audiences from diverse backgrounds.

# EDITOR'S NOTE

Dear Reader,

We, The Mathematics Society of St. Stephen's College, Delhi, are excited to present to you, our annual journal- Mathematica.

Publication of this issue brings us all extreme joy and satisfaction, as we depict an account of all the wonderful events and research projects that our junior members have undertaken, over the last academic year. The editorial board had a great time putting together this plethora of mathematical literature and memories of fun-filled and wonderful events of this past year. Receiving research articles from other sources, for publication in our journal, was a cherry on top, and it gives us immense joy that we were able to present them through our platform.

This edition also entails the research projects undertaken by our junior members through the prestigious Prof. Nagpaul Fellowship- a program which gave a platform to the students conducting research in diverse domains of Mathematics, ranging from graph theory and advanced algebra, to the application of Fourier and Gabor transform.

The research articles published in this edition effortlessly take us through the applications of mathematics in different areas of study. We begin by taking you through the wonders of 'Computational Chemistry' which helps us understand how a computer can now help us simulate real matter! Next, we try to establish a connection between the abstract and the applied side of this subject, by relating the concept of 'Probability' with 'Group Theory'.

We also dip our toe into Economics, which we know to be deeply related to Mathematics, by our prime focus on the 'Prisoner's Dilemma' using the concept of the prominently popular Nash Equilibrium. Lastly, our primal passion for our subject is brought out, as we take you through 'My two favourite algebraic inequalities', a beautiful presentation of the 'HM-GM-AM-QM Inequality'. All the articles published in this edition aim to illustrate a very distinct aspect of Mathematics.

The Mathematica Editorial Team extends a heartfelt thanks to the Department of Mathematics, at St. Stephen's College, Delhi for its unwavering support and assistance. We would like to thank all the contributors for their tireless efforts in adding to the knowledge in their respective fields. We would also like to thank everyone else who was directly or indirectly involved with publication of this journal for their efforts. Finally, we wish to express our gratitude to you, the readers, for picking up and reading this humble outcome of our united efforts and insatiable interest in Mathematics.

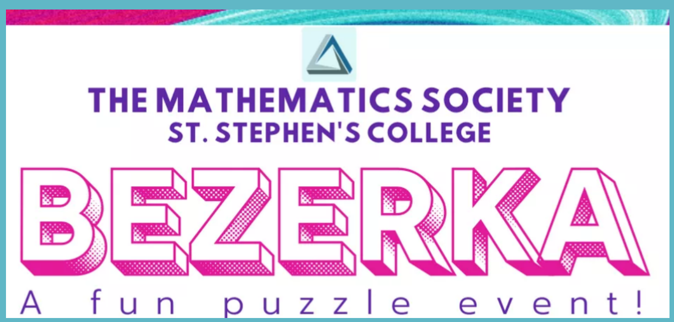
*Riya Garg*
*Editor-in-Chief*
*2021-22*

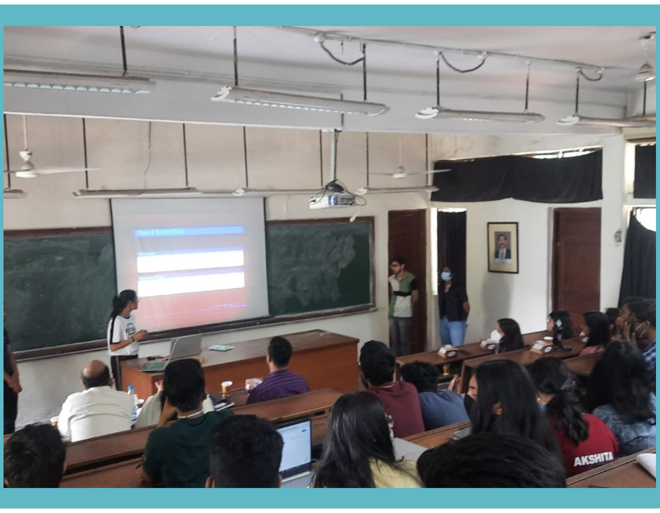
# OUR EVENTS

## ODD SEMESTER

### BEZERKA

This is a fun competition with logical questions, movie/song name guessing with mathematical symbols, and puzzles. It's a timed game in which players have five, ten, or fifteen minutes to complete each round.



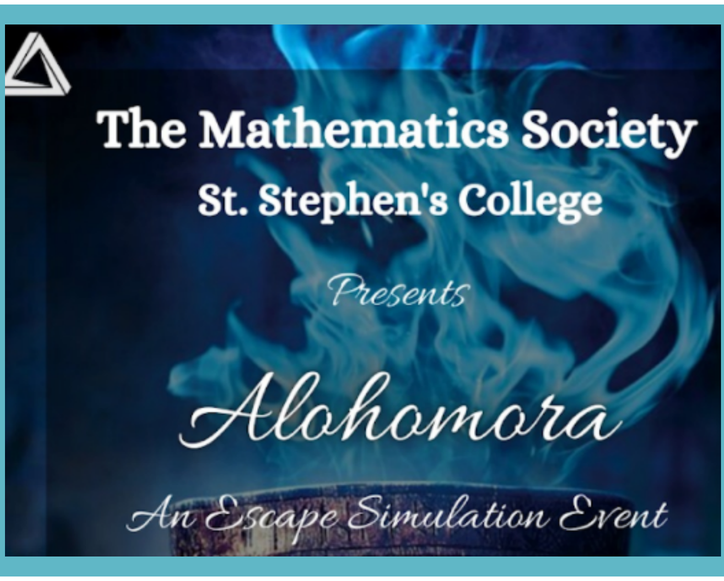
### PROFESSOR NAGPAUL FELLOWSHIP

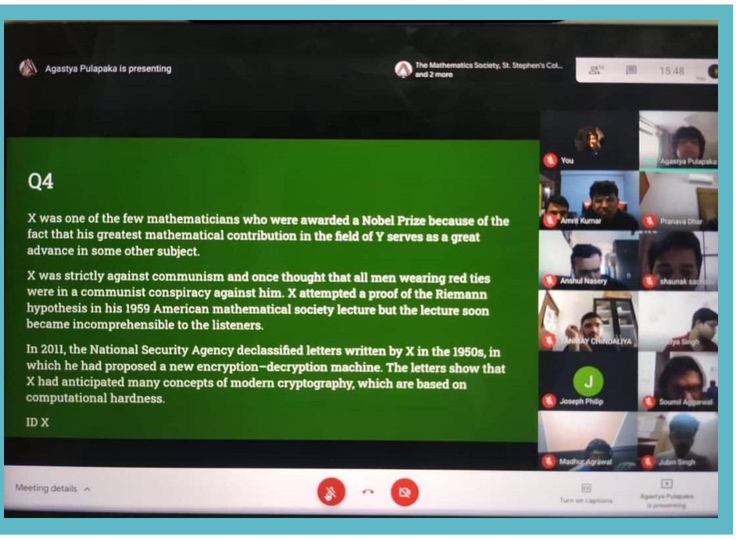
A collaboration between a group of students and a professor who work together to produce a research paper on a topic in Mathematics. The Professor Nagpaul Fellowship provides mathematically motivated students an opportunity to develop an interest in research as well as a chance to nurture their interest.

# ODD SEMESTER

## ALOHOMORA

A thrilling escape room simulation game based on the magical world of Harry Potter, the event is packed with adventure, and exit is contingent on making the right decision. It witnessed the participation of 1000+ people from various universities.




## MATHOPOLIS

This event, co-hosted by the Quiz Club, is a quiz on anything related to mathematics, from Leibniz to Lebesgue, Galois to Game Theory. For quizzers, it's a mental war, and for quizmasters, it's an opportunity to polish their talents. It consists of two rounds: The Prelims and The Finals.

# ODD SEMESTER

## X-TRA INNINGS

This was an event that brought together cricket and Math enthusiasts from different fields. 200+ students from different institutes across the country participated in the 3-day event. The overall theme of the event was IPL 2021. The preliminary round tested the participants on their mental ability, while further rounds included an auction round where team owners purchased players to create the best teams. Members of the winning team then competed against each other to win the competition.

# EVEN SEMESTER

## INTEGRATION

Integration is our annual fest consisting of fun-filled mathematical events, which brushes the participants' mathematical as well as analytical skills through diverse events Mathopolis, Guesstimate, Sicilian Feud, and Shakh-i-Mat Arena, along with a tinge of fun and creativity.

**Mathopolis 2.0** - A numbers quiz where anything related to numbers can be asked. From the 7 deadly sins to the 3 musketeers, participants have to be ready for the unexpected.

**Guesstimate** - A mass appealing event, where contenders use their Math and estimation skills.

**Sicilian Feud** - Sicilian Feud, a bidding contest that included everything from strategies to logic, and quizzes to case studies.

**Shakh-I-Mat Arena** - A completely new event this time, Shakh-I-Mat Arena is an online chess tournament where participants can show off their smartness.

**SymmetriX** - A test of participants' visuo-spatial abilities, this event required participants to search for real-life objects embedded in images.

# EVEN SEMESTER

## SYMPOSIUM

Symposium is a two-day series of lectures, workshops and tutorials on various topics related to Mathematics research. Professors from across the country deliver these lectures to an audience of more than 300 college students, including those from other countries. This year's theme was "Real Life Applications of Abstract Mathematical Concepts".

## PROF. SB MATHUR LECTURE SERIES

The Professor SB Mathur Memorial Lecture Series is a 2- day event which hosts eminent professors from institutes across Delhi to deliver lectures on various branches of Mathematics. New and innovative topics are taken up every year so as to increase awareness and diversity of the discipline.

# EVEN SEMESTER

## TREASURE HUNT

The flagship event of the Mathematics Society, the Treasure hunt is an inter-college biennial competition. It is one of our biggest events and high amounts of managerial planning and working efficiency goes into the process. Every team is provided clues that direct them to the following team's location. It is an entertaining event in which participants are exposed to math. To make the event more interesting, it was based on Pirates theme this year.

# RESEARCHES BY JUNIOR MEMBERS

## Max Flow Min Cut algorithm and its applications

We studied the topic, "Graph Theory and Network Flow Problems". We were particularly working on Max Flow Min Cut algorithm and its application. It comprised of various algorithms like Dijkstras Algorithm, Ford Fulkerson, Kruskal Algorithm, Hungarian algorithm, Bipartite matching and their application in real-world as edge-disjoint paths in a graph, kidney market, minimizing production costs in an organization.
The fellowship this year was online but with the constant support and motivation from our guide, we were able to complete our paper. She helped us a great deal in finalizing our topic for the presentation and also in learning much more about the topic by providing us with the future prospects of it.
My experience has been great and fascinating throughout the course of this fellowship.
I would also like to thank The Mathematics Society and The Department of Mathematics, St Stephen's College for providing us with an opportunity to develop our research skills.

Hitesh Kumar and Bhanu Jain
Guide: Dr. Rajni Gupta
Nagpaul Fellowship 2020-21

# Applications of Fourier and Gabor Transform

I have always been fascinated by mathematics and my urge to learn more and more about the subject has been increasing ever since. It was in my 1st year when I got to know about the Nagpaul fellowship programme offered by our college. I immediately decided to try my luck and give my name for it but I soon found out it is not meant for first year students.
When the online second year came, I was somewhat confused about whether it would happen or not, but to my delight, It did happen and when I saw the topic, *"Applications of Fourier and Gabor Transform",* I knew I wanted to apply for it. I got an incredibly supportive guide in the form of *Mr. Piyush Bansal,* whom I would like to thank with all my heart.

My entire experience has been full of fascinating learning. Though it was a first-hand experience in the field of research, I never felt out of place or monotonous. It was after a long process of searching and information gathering that I finally decided to target my paper on the integration of Fourier Transforms and Statistics, two seemingly unrelated areas of mathematics. What I found was an incredibly beautiful proof of *The Central Limit Theorem* in statistics using Fourier Transform. I am extremely thankful to my partner, Trisha Debnath (who herself presented incredible research on *Applications of Fourier Transforms in Astronomy*) for all the help that she provided. Together, we presented our research titled *"Applications of Fourier Transform in Statistics and Astronomy".* All in all, it was a great project that I had had the opportunity to be a part of.

I would like to thank The Mathematics Society and The Department of Mathematics, St. Stephen's College for providing me with a chance to know what research and review are all about. The programme gives the budding minds the necessary impetus that they need and introduces research as a bright career option.

Athak Kumar Singh
Guide: Mr. Piyush Bansal
Nagpaul Fellowship 2020-21

# Applications of Fourier and Gabor Transform

A Physicist can never appreciate the role of Mathematics enough in a lifetime. How most of the physical interpretations are answered by mathematical formulations has enticed me since day 1 of this journey. Starting from the basic applications of Calculus in Mechanics to rigorous Linear Algebra in Quantum Mechanics, I have only evolved to understand how deep and beautiful the relationship between Physics and Mathematics is. What has always fascinated me the most is how the transformation from one space to the other yields fruitful results. Fourier Transform is an inevitable tool and has given legendary outcomes like Heisenberg's Uncertainty Principle, Magnetic Resonance Imaging, etc. So when I learned about the prestigious Professor Nagpaul Fellowship 2020 having an opening for "Applications of Fourier and Gabor Transform", I saw my urge of realizing that Mathematics (deeply) was finally going to have a formal platform.
Before I begin with the experience, I would like to thank my guide, Mr. Piyush Bansal, for being supportive in every possible manner throughout. Be it with the doubts or long preparatory calls for the final presentation, Sir has been patient throughout. With all resources from him, I began to work on the applications of Fourier transform in Astronomy. Isn't it simply beautiful how radio signals from millions of miles, all of a sudden speak a lot in frequency space? It's the Fourier transform that creates this magic! We are social animals and without the support of a peer, long days of theoretical preparations, reading, and solving problems wouldn't have been this memorable. My partner, Athak Kumar Singh has been an amazing one since day one. In a sense, this fellowship has also strengthened that senior-junior bond.

Finally, we presented our work on "Applications of Fourier transform in Statistics and Astronomy" after months of work and google meet sessions. Experience of live preparation and presentations would have hit different, but the online one was worth the hassle.
To conclude, it was an amazing learning experience. A warm thanks to the Mathematics Society and Department of Mathematics of St. Stephen's College for this wonderful opportunity. All in all, this Fellowship has definitely helped me to develop my research and presentation skills. It is an excellent platform for the able minds in college, to absorb and to radiate.

Trisha Debnath
Guide: Mr. Piyush Bansal
Nagpaul Fellowship 2020-21

# Graph Theory and Network Flow Problems

I am Akanksha (BSc. Programme with Computer Science) and I pursued Professor Nagpaul Fellowship 2020-21 under the sincere guidance of Professor Rajni Gupta. My group members were Harsh Bindal, Bhanu, and Hitesh. We studied the topic, “Graph Theory and Network Flow Problems”. Harsh and I particularly explored the topics of vertex coloring, handwriting recognition, and traffic light control at crossroads and wrote a research paper together.

Although the fellowship was online, still it was a delightful experience to explore the topics, not in our course. Ms. Rajni Gupta, our guide and mentor, passionately helped and motivated us to study the depths of the subject. The meetings that were organized twice a month with our group members and our mentor were very informative. Not only presenting our topics was joyful but also watching the presentations of other members was quite interesting. We worked with full dedication and tried to improve our knowledge and skills during the fellowship.

We got an opportunity to learn more from the presentations of other groups on the day of the final presentations. Overall, it was an amazing and informative experience.

Akanksha Sharma and Harsh Bindal
Guide: Dr. Rajni Gupta
Nagpaul Fellowship 2020-21

# Advanced Algebra

The Professor Nagpaul Fellowship organized by the Mathematics Society of St. Stephen's for the year 2020-21 was a journey into new areas of math. For my partner and I, being selected to explore advanced topics in Algebra, with an emphasis on mathematical software, we began our fellowship by learning how to use SageMath (a free, open-source alternative to Mathematica, Matlab, and Maple). Specifically, within SageMath, we got comfortable with using the GAP subsystem for working with groups.

Our advisor steered us towards studying the concept of Symmetry, in both 2 and 3 dimensions, a topic that had been mentioned in our Group Theory paper that semester. We began by understanding the background of the concept, the concepts of isometries in the Euclidean plane, and then used that knowledge to further study what symmetry groups exist in the Euclidean plane (both finite groups and infinite). From there we got to how one could begin to classify various patterns that could be observed in the world, in nature, art, and architecture, because these patterns are merely artistic expressions of the infinite symmetry groups in a plane.

Our next course of action was to study symmetry in 3 dimensions, particularly in the context of the Platonic solids. We learned to establish a rigorous definition for these solids (convex regular polyhedra). The 5 solids (tetrahedron, cube, octahedron, dodecahedron, and icosahedron) allowed us to explore the concepts of a dual solid, and in particular, for the cube and octahedron, understand the concept of inversions (a type of isometry in 3D space)

We cannot thank our mentor enough for all her guidance, and we're looking forward to the idea of exploring mathematics beyond the classroom.

Lael John and Shayanak Kundu
Guide: Dr. Jaspreet Kaur
Nagpaul Fellowship 2020-21

# Applications of Abstract Mathematical Structures

Jatin Talreja and I were research partners, working under the guidance of Mr. Kashif Ahmed, for the Nagpaul Fellowship, 2021. This topic was particularly compelling because of my fascination with the scope of abstract mathematics. I consider myself fortunate for having a research guide who augmented this interest of mine with his support, patience, and encouragement. It has often been said that "The best teachers are those who show you where to look, but don't tell you what to see" but I truly realized the truth of this statement when Kashif Sir gave us free rein to explore the subject. During the initial exploratory phase, we spent lots of time studying the applications of modular arithmetic and Galois fields in cryptographic systems while simultaneously exploring topics of visual group theory like frieze groups, wallpaper groups, and braid groups. Both areas were riveting to us and demanded further study, and thus we found ourselves in a dilemma. Finally, this indecision impelled an idea, we decided to demonstrate the intersectionality of the two fields using an original algorithm. We applied the theory of braid groups to translate the cryptographic techniques used in the algorithm of the "one-time pad". It was a unique experience because it was the perfect amalgamation of thorough study and brainstorming, ideal for anyone looking to satiate their scholastic vigor as well as exercise independent thought. Nagpaul Fellowship gives us an occasion to think analytically and innovate;
moreover, the insights and feedback from research guides expedite the gain of academic maturity. As undergraduate students, the notion of research is unfamiliar and consequently intimidating to most of us; hence, opportunities like the Nagpaul fellowship give us valuable exposure to the field and help ease us into it.

Srinwanti Debgupta and Jatin Talreja
Guide: Mr. Kashif Ahmed
Nagpaul Fellowship 2020-21

# Modelling COVID-19 Using R

The topic of our group's study was COVID 19 using R. We started off with some sessions covering the basics of R. After successful completion of the sessions, we all were given some research papers to look at. On finalizing the paper, we created a framework covering the Epidemic Models and incorporated the one covered in the finalized paper along with a hypothesis to check the genuineness of a Media Claim. After getting the approval from our guide, it was presented on 25th April 2020 during the Final Presentations of the Nagpaul Fellowship.
The two main skills that we were able to develop through this paper were:
1. Exploratory Analysis
2. Reproducible Research Conclusion.

Through the mathematical models, we got a general idea of how the various compartments behave over a period of time ideally. From the data we used for Delhi, we found out that in the case of the infectives and recovered compartment, the general trend is almost the same, as the one we got from the SEIR model, although the nature of the curve is a little different.

The Fellowship provided unique insight into the process of developing a research paper. We learned about the importance of teamwork and attention to detail. Overall, it was a rewarding experience with a lot to take away from. We are extremely grateful to our guide and mentor, Ms. Rajni Gupta for her constant support and guidance throughout the course of the paper and to The Mathematics Society for giving us this opportunity.

Marwin Solomon, Shravya Ann George, Keerti Panwar
Guide: Dr. Rajni Gupta
Nagpaul Fellowship 2020-21

# RESEARCH ARTICLES

## TEACHING MOLECULES TO COMPUTERS

- ANSON THOMAS (B.SC. (H) CHEMISTRY
ST. STEPHEN'S COLLEGE,
2017-2020)

The primary motivation to do science is to develop a deep understanding of the world around us. Scientists and laymen alike have often resorted to experiments to understand the nature of materials around them. All this led to the development of various branches of science, including chemistry, physics, engineering, to name a few. Computational science is one such branch that uses computer technology to assist in predictions, calculations, and simulations. These tools can predict the results, which can then be tested experimentally. Computational Chemistry helps save time, chemicals, exposure to chemicals, and much more! We would try to understand how a computer, running some software, can simulate real matter.

We begin our story from the 18th century when Dalton gives his atomic theory. It was primitive and had many flaws. Improving upon Dalton's theory, many other theories were also proposed. A milestone in understanding the structure of the atom came with the Rutherford experiment. Based on this experiment, Rutherford proposed his Planetary model of atoms. He likened the atomic system to the solar system, where the electrons, like planets orbit around the sun, which happened to be the positively charged nucleus. However, it was countered by the classical electromagnetic theory. According to this, a charged particle, like an electron, when accelerated, would lose energy continuously. Eventually, it would lead to the collapse of an atom. This is clearly not the case, as atoms are known to be stable. Then Bohr came along who said that negatively charged electrons revolved around the positively charged nucleus with a definite quantum of energy in well-defined orbits in an atom. In other words, the energy of the electron in an atom was quantized. This concept of quantization was compelling and paved the way for modern quantum mechanics. He said that when an electron made a jump from one orbit to another, it either absorbed energy or released it. He also gave the formulae for the radius, energy of the atom, and energy of wavelength released.

de-Broglie introduced the next central concept, which is the dual nature of matter. Because of this, the electron, which was being considered classically until now, could also be treated as a wave. Heisenberg later pointed out that this dual nature of matter/energy would naturally limit our ability to measure a microscopic entity's position and momentum simultaneously with good precision.

With this background, we step into the realm of quantum mechanics (QM). A wavefunction is used to represent the state of a system in QM. Most commonly, the wave function is represented using psi($\psi$) and is a function of the time and position of that system. It contains all the properties of a system that one wants to study. The more refined the wavefunction is (i.e., takes into account electron correlations, excited states, etc.), the better is the prediction of properties. However, by itself, psi does not give much information because it has no physical meaning!

Various operations are performed on psi to extract information about the system. Squaring psi, for instance, gives the probability density of finding the electron at the point. Multiplying this by the volume density gives the probability of finding the electron in a given space. To get more properties of the system from psi, we take the help of operators. The operator operates on the wavefunction and returns the value of the physical property we are interested in along with the operator. There are operators for the position, linear and angular momentum, kinetic energy, etc.

If you haven't guessed it by now, then yes, operators operating on the wavefunction is an eigenvalue problem. The operators operate on the eigenfunction (wavefunction) to return the eigenvalue, the physical property (say, kinetic energy, angular momentum, etc.) we are looking for, and the eigenfunction back.

The most famous among these operators is the Hamiltonian operator, which gives the system's total energy. Hamiltonian is used in the Schrodinger wave equations (time-dependent and independent), which form the basis for all quantum mechanics, and by extension, all computational chemistry and related fields. Given below are time-dependent and independent SWEs.

$$\hat{H}\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t} \text{ (time-dependent SWE)}$$

$$\hat{H}\psi(t) = E\psi(t) \text{ (time-independent SWE)}$$

Unfortunately, the Schrodinger wave equation can only be solved for a handful of systems, like particle in 1D (2D and 3D) box, Rigid rotor, and the harmonic oscillator. Hydrogen atom (and H atom-like system) is the last and only real system for which SWE can be solved completely. The Hamiltonian operator for a hydrogen atom is composed of the terms describing the electron's kinetic energy (KE) and its potential energy (PE).

$$\hat{H} = \widehat{KE} + \widehat{PE}$$

$$\hat{H} = -\frac{\hbar^2}{2m_e}\nabla^2 - \frac{Ze^2}{r}$$

$m_e$ is the electron's mass, and **r** is the distance between the electron and the proton.

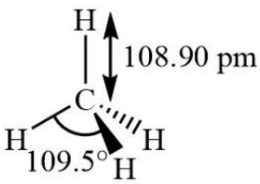
For multielectronic, and multibody problems, the Hamiltonian operator becomes quite complex, and therefore, we employ approximations to get the eigenvalues. This is illustrated in the example below.

Let's say we want to model a methane molecule (shown in the picture). To communicate this to a computer, we use an input file. This has the positions of all the atoms of the molecule, along with additional information such as bond lengths, atomic numbers, etc. Usually, this is given in the form of Cartesian coordinates.

For small systems, there is an alternative, more convenient representation called Z matrix representation. Unlike the Cartesian system, where the coordinates of all atoms are written explicitly, in a Z matrix representation, the position of one atom is defined in space. Other atoms are defined with respect to this atom. Both the representations for a methane molecule has been shown below:

**Cartesian system**

| Atom | X (pm) | Y (pm) | Z (pm) |
|---|---|---|---|
| C1 | 0.000000 | 0.000000 | 0.000000 |
| H2 | 0.000000 | 0.000000 | 108.9000 |
| H3 | 102.6719 | 0.000000 | -36.3000 |
| H4 | 51.3360 | -88.9165 | -36.3000 |
| H5 | 51.3360 | 88.9165 | 36.3000 |




**Z matrix**

| Atom | Distance (pm) | Bond angle | Torsional angle |
|---|---|---|---|
| C1 | | | |
| H2 | 1 108.9000 | | |
| H3 | 1 108.9000 | 2 109.4710° | |
| H4 | 1 108.9000 | 2 109.4710° | 3 120.000° |
| H5 | 1 108.9000 | 2 109.4710° | 3 -120.000° |

The atomic information (atomic mass, number of electrons) could be already present in the software (such as Gaussian, Gamess etc.) or provided by extra files (called pseudopotentials) in softwares like VASP, Quantum Espresso. With the input file at hand, our program has 2 entities mainly to deal with – electrons and the nuclei of the atoms. Thanks to Born and Oppenheimer, they can be treated independently, making our lives much simpler! They gave an approximation, commonly called the Born-Oppenheimer approximation, which says that since the nuclei are much heavier than the electrons, their motions can be treated independently. In simpler terms, their energies are independent of each other. This is exploited to solve the SWE approximately.

First, the nuclei are assumed to be fixed (an approximation since nuclei move much slower than electrons). This helps to write the Hamiltonian for the electronic part (electronic Hamiltonian).

Total Hamiltonian for a system with N electrons and M nuclei can be written as:

$$\left[-\sum_{i}\frac{\hbar^2}{2m_e}\nabla_i^2-\sum_{I}\frac{\hbar^2}{2M_I}\nabla_I^2+\frac{1}{2}\sum_{i\neq j}\frac{e^2}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}_i-\mathbf{r}_j|}\right.$$
$$\left.+\frac{1}{2}\sum_{I\neq J}\frac{e^2}{4\pi\epsilon_0}\frac{Z_IZ_J}{|\mathbf{R}_I-\mathbf{R}_J|}-\sum_{i,I}\frac{e^2}{4\pi\epsilon_0}\frac{Z_I}{|\mathbf{r}_i-\mathbf{R}_I|}\right]\Psi=E_{\text{tot}}\Psi$$

Here,

$$-\sum_{i=1}^{N}\frac{\hbar^2}{2m_e}\nabla_i^2-\sum_{I=1}^{M}\frac{\hbar^2}{2M_I}\nabla_I^2$$ represents the kinetic energy of electrons and nuclei

$$\frac{1}{2}\sum_{i\neq j}\frac{e^2}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}_i-\mathbf{r}_j|}$$ potential energy because of electron-electron repulsions

$$\frac{1}{2}\sum_{I\neq J}\frac{e^2}{4\pi\epsilon_0}\frac{Z_I Z_J}{|\mathbf{R}_I-\mathbf{R}_J|}$$ potential energy because of nucleus-nucleus repulsion

$$-\sum_{i,I}\frac{e^2}{4\pi\epsilon_0}\frac{Z_I}{|\mathbf{r}_i-\mathbf{R}_I|}$$ potential energy because of electron-nucleus attraction ≡ $\sum_i V_{\mathrm{n}}(\mathbf{r}_i)$

Since we are only dealing with electrons, nuclear terms vanish, and the electronic terms remain, giving the electronic Hamiltonian.

$$\left[-\sum_i\frac{\nabla_i^2}{2}+\sum_i V_{\mathrm{n}}(\mathbf{r}_i)+\frac{1}{2}\sum_{i\neq j}\frac{1}{|\mathbf{r}_i-\mathbf{r}_j|}\right]\Psi=E\,\Psi$$

The solution to this SWE gives the electronic wave function that describes the motion of the electrons. The eigenvalue corresponds to the electronic energy. The electron densities, n(r), are used to determine the electrostatic field generated by the electrons, VH(r). This field is plugged back into the SWE (all the while ignoring electron-electron interactions) to get the new wavefunction, and the calculations are repeated to obtain the new energy.

$$\left[-\frac{1}{2}\nabla^2 + V_{\mathrm{n}}(\mathbf{r})\right]\phi_i(\mathbf{r}) = \varepsilon_i \phi_i(\mathbf{r})$$

$$n(\mathbf{r}) = \sum_i |\phi_i(\mathbf{r})|^2$$

$$V_{\mathrm{H}}(\mathbf{r}) = \int d\mathbf{r}' \frac{n(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$\left[-\frac{1}{2}\nabla^2 + V_{\mathrm{n}}(\mathbf{r}) + V_{\mathrm{H}}(\mathbf{r})\right]\phi_i^{\mathrm{new}}(\mathbf{r}) = \varepsilon_i^{\mathrm{new}} \phi_i^{\mathrm{new}}(\mathbf{r})$$

Calculations are repeated until the change in energy is less than the threshold value. This is Hartree's self-consistent field approximation (1928) and is of pivotal importance in computational chemistry. It is not perfect by any means, and many improvements have been made. Several techniques, such as Density Functional theory, molecular dynamics, etc., have been developed, which give values closer to experiments. However, this cycle of iteratively solving the SWE and getting the eigenvalues is at the heart of all of these.

**References**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# CONNECTION OF PROBABILITY WITH GROUP THEORY

- ATHAK KUMAR SINGH (B.SC. (H) MATHEMATICS,
ST. STEPHEN'S COLLEGE,
2019-2022)

Mathematics has always been and will always continue to be the queen of all sciences. It is marvellous in the fact that it is both a form of science and a form of art. The beauty of mathematics can only be appreciated by those who do it.

For the sake of better understanding and systematic study, mathematics is often divided into branches/fields. It often creates a misconception in the minds of readers and students that these fields are completely disjoint from each other. This, however is not true. No field of maths is exclusive. There are intricate, beautiful and often hidden connections between different areas of mathematics and this article aims to prove the same by establishing relations between two different fields of maths, namely Group Theory and Probability Theory.

We shall establish a result known before but not as famous as the results of Group Theory are:-

Result:- In a cyclic Group G =<a> the probability that a given element a^k is a generator of G does not depend on order of G, say n. It only depends on π(n), the positive prime factors of n. Before we proceed with the proof, we shall make use of a lemma:-

**Lemma**:–
$\phi(p^k) = p^k - p^{k-1} = p^k(1-\frac{1}{p})$, where $p$ is a prime number, $k \in \mathbf{N}$ and $\phi$ is Euler's Totient Function, also known as Euler's Phi function
**Proof** –
By the definition of $\phi$, $\phi(n)$=no. of positive integers less than n, which are co-prime with n.
Therefore, $\phi(p^k)$=no. of positive integers less than $p^k$, co-prime with it.
Now, Since p is a prime, the only proper divisors of $p$ are $p, 2p, 3p, \ldots\ldots, p^{k-1}p$ which are $p^{k-1}$ in number. For every other positive integer $m$, $gcd(p^k, m) = 1$
Therefore, $\phi(p^k) = p^k - p^{k-1} = p^k(1-\frac{1}{p})$

Now, the proof of the main result, Let G =<a> be a cyclic group with one of its generators as a. Let order of G = o(G) = n.
Then, since o(G) = n and no. of generators=φ(o(G)), the probability that a given element of G, is a generator of G is given by

$$P(g) = \frac{\text{no. of generators of } G}{\text{no. of elements in } G} = \frac{\phi(n)}{n}$$

Let π(n)=set of all positive prime factors of n = . $\{p_1, p_2, \ldots, p_m\}$
By the Fundamental Theorem of Arithmetic,
$n = p_1^{q_1} p_2^{q_2} \ldots p_m^{q_m}$ , for some positive integers $q_1, q_2, \ldots, q_m$ .

Therefore,

$$P(g) = \frac{\phi(n)}{n} = \frac{(p_1^{q_1}(1-\frac{1}{p_1})p_2^{q_2}(1-\frac{1}{p_2})\ldots p_m^{q_m}(1-\frac{1}{p_m}))}{p_1^{q_1}p_2^{q_2}\ldots p_m^{q_m}} = (1-\frac{1}{p_1})(1-\frac{1}{p_2})\ldots(1-\frac{1}{p_m})$$

which is independent of n and depends only on π(n). Thus, we have proved what we aimed for.

The result can be illustrated using various examples:-

**1**. Let $G = \mathbf{Z}_7$ under the operation addition modulo 7
Then, o(G)=7
$\pi(7) = 7$
Generators of G=1,2,3,4,5,6
$P(g) = \frac{6}{7} = (1 - \frac{1}{7})$

**2**. Let $G = \mathbf{Z}_{12}$ under the operation addition modulo 12
Then, o(G)=12
$\pi(12) = 2, 3$
Generators of G=1,5,7,11
$P(g) = \frac{4}{12} = \frac{1}{3} = (1 - \frac{1}{2})(1 - \frac{1}{3})$

**3**. Let $G = \mathbf{Z}_{24}$ under the operation addition modulo 24
Then, o(G)=24
$\pi(12) = 2, 3$
Generators of G=1,5,7,11,13,17,19,23
$P(g) = \frac{8}{24} = \frac{1}{3} = (1 - \frac{1}{2})(1 - \frac{1}{3})$

Note- From examples 2 and 3, we can notice that both $\mathbf{Z}_{12}$ and $\mathbf{Z}_{24}$ have the same prime factors although their orders are different. Hence, they have the same probability of finding a generator from their elements.

Thus, we conclude our article by saying that as more and more discoveries take place, mathematics evolves and intimate connections are established be tween different fields of study which are otherwise considered exclusive. Math ematics is like a whole new universe consisting of various celestial bodies with everything linked together by basic forces of nature. More than anything else, it contains vacuum which is waiting to be filled by the genius of some human.

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# PRISONERS' DILEMMA AND NASH EQUILIBRIUM

- RHEA AGARWAL (B.SC. (H) ECONOMICS, RAMANUJAN COLLEGE, 2020-2023) -

The Hollywood movie 'The Beautiful Mind' might provides a fair idea about John Nash, who gave the Game Theory. The literal definition of the Game Theory from Princeton is 'a stable state of system that involves several interacting participants in which no participant can gain by a change of strategy as long as the other participants remain unchanged'. Let us understand this through an example.

One day, the police made two apparently unrelated arrests of Jack and Ben. They brought them separately to the police station and told them that this was an open and shut case. Little did they know that these were the two people that actually committed a much more serious offense of armed robbery a few weeks ago. However, there was no concrete evidence. So they want to give them an incentive to snitch on each other. They told the criminals that they are going to get five years of imprisonment, which is guaranteed. But if Jack confesses his crime, while Ben does not, Jack gets away scot free while Ben gets ten years and vice versa. If both of them confess and cooperate with the police, they can get away with five years each. The third case is that if both refuse to confess they both get three years of imprisonment. We will assume that the prisoners do not have a loyalty pact with other, so they do not have any reason to trust each other. This scenario is called Prisoner's Dilemma. So will they remain loyal to each or sell each other out? What method will they adopt to arrive at a conclusion? Let us find out through a payoff matrix.

Here, Nash Equilibrium comes into picture. Recall that Nash Equilibrium occurs when no participant can gain by a change of strategy as long as the other participants remain unchanged. Refer to the payoff matrix for easier understanding.

1. Let us say that we are in block 1, where both Jack and Ben are denying. Now, moving from block 1 to 2, where Jack is constant (meaning that he is always denying), Ben gains from 3 years of imprisonment to 0 years. When we move from block 1 to 3, we have Bill constant as he is denying in both the cases, Jack can improve his situation from 3 years to 0 years of imprisonment. Hence, it is not a Nash Equilibrium
2. When we move from block 2 to block 1, Jack is constant (always denying). Ben will not be benefited here by a change of strategy from confession to denial because he will have to face3 years of imprisonment instead of 0 years. So far it conforms to Nash Equilibrium. However, when we move towards block 4, we have Ben constant, who is always confessing. Here Jack can improve his situation by from denying to confessing as he will ultimately face 5 years of imprisonment instead of 10 years. Hence, it is not a Nash Equilibrium.
3. Now let`s move from block 3 to block 1, Ben is constant (denying). Jack will have to face 3 years instead of 0 years, if he changes his strategy by moving from confession to denial. When move towards block 4, Jack is constant (confessing). So if Ben changes his strategy from denying to confessing, he will have to suffer 5 years instead of 10. Since he can improve his situation, it is not a Nash Equilibrium.
4. Finally we are in block 4 and can either move to block 2 or block 3. From block 4 to 2, Ben is constant (confessing), but Jack cannot be advantaged by change of strategy from confessing to denying as his imprisonment increases to 10 years. In the second case, while moving to block 3, Jack is constant (confessing). Here, Ben is not benefitted because if changes his strategy from confession to denial, he will be sentenced to 10 years of imprisonment instead of 5 years. Since, the convicts cannot gain in any condition in block 4, it is Nash Equilibrium.

This implies that it is safe for both of them to confess as this the most ideal situation, given that they do not know the decision of the other one. Taking example of Ben, if Jack denies committing the crime, he gets away scot free. Even if Ben and Jack both confess, Ben's imprisonment remains at 5 years only. Same is true from Jack's perspective.
Now let us use the game theory to understand why firms in a cartel cheat. A cartel is an unlawful agreement between players of oligopoly. To start with, let us go through some integral economic concepts.
Marginal cost (MC) is defined as the additional cost incurred in the production of one more unit of a good or service. Market marginal cost is the marginal cost for all firms in the market. Marginal revenue (MR) is the increase in revenue that results from the sale of one additional unit of output. Market marginal revenue is the marginal revenue of all firms in the market. While marginal revenue remains constant over a certain level of output, it follows law of diminishing returns and will eventually slow down as the output level increases.

Firms continue producing until their MC=MR. Average cost is the total cost divided by number of units produced. Average total cost is defined as the sum of fixed cost and variable cost divided by total units produced. Market demand is the sum of individual demand for a product from all the buyers in the market. We are going to assume that both of these firms are acting together. If they perfectly coordinate, they can join their capacities and act like a monopoly. Hence, the individual curves for a firm has twice the slope of the corresponding market curve in case of a duopoly.

The firms will keep on producing till point M as it is the optimal quantity. Before that the firms have a profit as MR< MC. After point M, if the firms produce, MC> MR and the firms incur economic losses. The price they will get for producing 50 units as represented by the graph. The price they would get for producing 50 units is given by Market demand curve which is ₹A per unit (Market demand curve gives the price). So, the profit per unit is given by BC (Market ATC– Market demand). So their total economic profit is the area ABCD (quantity produced multiplied with profits per unit) if they behave coordinate perfectly essentially like a monopoly. Now let us say that BC is ₹10, so the total economic profits of both the firms in cartel = 50*10= ₹500.

Both the firms in the cartel agreed that they will produce half the units i.e. 25 units each and also split the profits. Now they produce 25 units each which leads to an economic profit of FE per unit which is ₹10 per unit. This leads to a total economic profit for each individual firm as 25* 10=₹250 units or price per unit* total units produced. This area is represented by red area in the figure. This is equal to half of the total economic profit of both the firms combined.

Now one firm in the cartel comes up with an idea that it is not going to produce 25 units. In other words it wants to breach the contract. It wishes to produce 35 units. So, the total quantity in the market is now 60 units.

The market demand curve which gives the price is now ₹G per unit and the economic profit per unit is given by Market demand or price- Market ATC i.e. line HI on the graph. So, the total economic profits for the firms= region GHID (quantity* economic profits per unit). This is represented by the blue region in the graph. Let's say length of HI= 8 units, so numerically, the economic profits of both the firms in the cartel= 8*60= ₹480. Notice that total economic profits decreased from ₹500 (ideal scenario) to ₹480 (in which one firm cheats). This makes sense because now the market is producing more than the equilibrium quantity at which Market MC= Market MR. So, they grey region in the graph represents the economic loss incurred of ₹20 due to breach of contract by one firm. Market MC is higher than the market revenue at all points beyond the equilibrium point. But, even in this case the cheating firm manages to have profits. Due to producing 35 units (instead of 25) at a profit of ₹8 per unit, it manages a total economic profit of 35*8= ₹280. The honest firm has to settle down with a total economic profit of 25*8= ₹200.
While the cheating firm could increase its economic profits from ₹250 to ₹280 by producing 10 units more than promised, the total economic profits of the honest firm decreased from ₹250 to ₹200.

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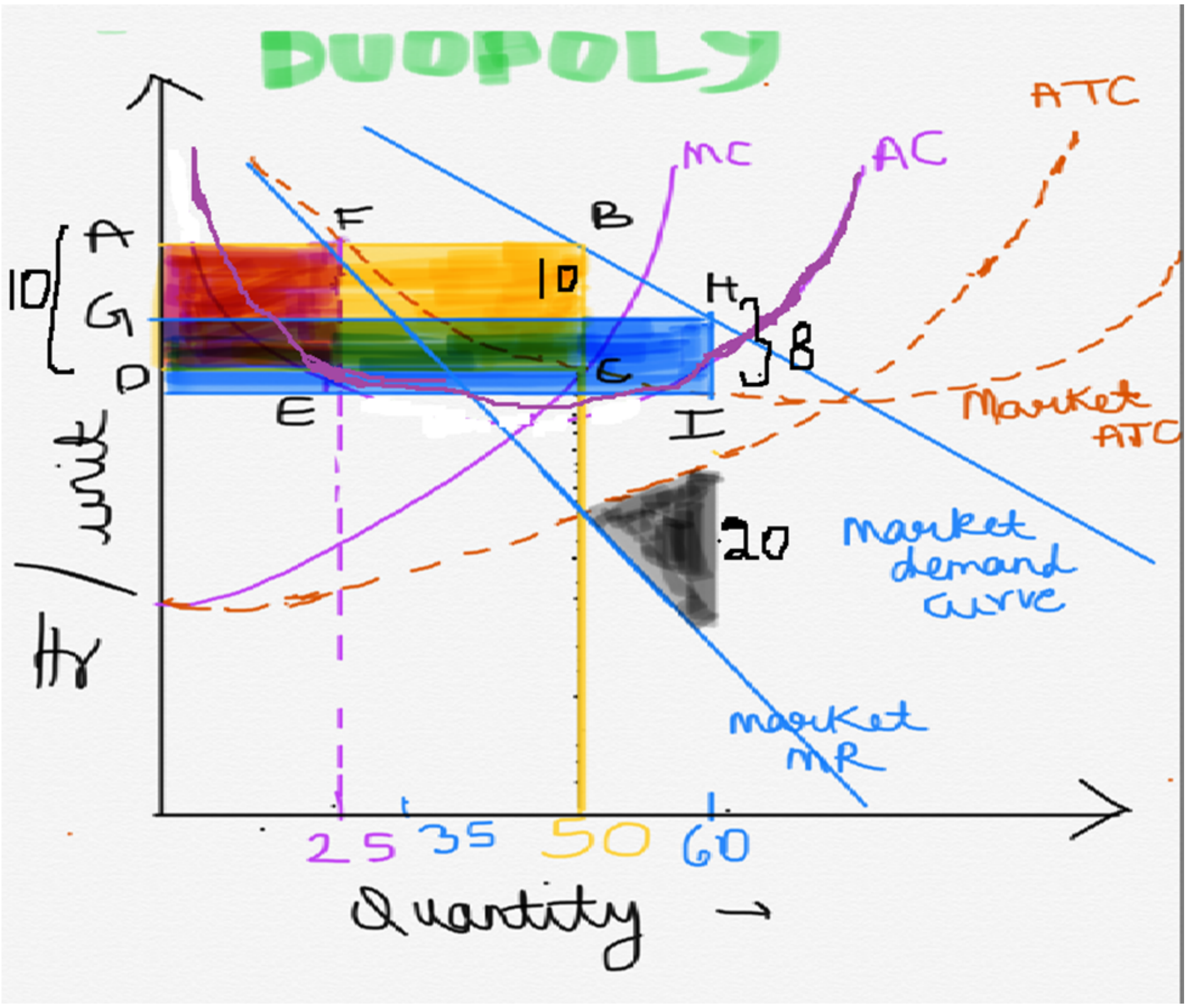


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# MY TWO FAVORITE ALGEBRAIC INEQUALITIES

--JIBRAN IQBAL

**Theorem HM-GM-AM-QM inequality : –**

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}} \le \sqrt[n]{x_1 \cdot x_2 \cdots x_n} \le \frac{x_1+x_2+\cdots+x_n}{n} \le \sqrt{\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}}$$

For all positive real $x_i$ with equality at $x_1 = x_2 = \cdots x_n$

**Proof : (by Cauchy)**

Cauchy used induction in his proof. Now, in this article I'll try show you what may have been Cauchy's thought process to proving this using induction. Firstly, an inequality we all may be familiar with(this one is shown in high schools usually) is

$$(\sqrt{x}-\sqrt{y})^2 \ge 0 \Rightarrow x+y \ge 2\sqrt{xy} \Rightarrow \frac{x+y}{2} \ge \sqrt{xy}$$

This means that the mean of any two **positive** numbers (as we started out with $\sqrt{x}$ and $\sqrt{y}$) must be greater than (or equal to) the square root of its product. Let's try to use this inequality now and see what else we can derive. Let's try using this inequality on the expression $\frac{x+y}{2}+\frac{z+w}{2}$

$$\frac{\frac{x+y}{2}+\frac{z+w}{2}}{2} \ge \sqrt{\left(\frac{x+y}{2}\right)\left(\frac{z+w}{2}\right)}$$

Now, we can use the inequality we obtained for $\frac{x+y}{2}$ and similary for $\frac{z+w}{2}$ to get

$$\sqrt{\left(\frac{x+y}{2}\right)\left(\frac{z+w}{2}\right)} \ge \sqrt{\sqrt{xy}\sqrt{zw}} = \sqrt[4]{xyzw}$$

$$\frac{x+y+z+w}{4} \ge \sqrt[4]{xyzw}$$

We start to see a pattern here now, but we've only seen 2 cases so it's a bit too early to make a reasonable guess. Let's use the inequality

$$x^2 + y^2 + z^2 - xy - yz - zx \geq 0$$

A proof of this is given in the first question of the "Introductory inequalities" section. Now, let's say that $x, y, z \geq 0$ which means we're limiting ourselves to positive reals only for now. This means that $x + y + z \geq 0$, so let's multiply it with the inequality above

$$(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) \geq 0$$

Expanding (painfully) and simplifying, we get

$$x^3 + y^3 + z^3 \geq 3xyz$$

If we consider the substitution $x \to \sqrt[3]{a}$, $y \to \sqrt[3]{b}$, $z \to \sqrt[3]{c}$

$$\frac{a + b + c}{3} \geq \sqrt[3]{abc}$$

Now we get to see a pattern, maybe this inequality holds for $n$ variables? rather than just 2,3 and 4. Well, we can always use induction to prove something like this. We'll use **Forward-Backward Induction** and basically show that $P(n) \Rightarrow P(2n)$ and $P(n) \Rightarrow P(n-1)$.

We want to prove the statement

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$$

for $a_i \in \mathbb{R}^+$ with equality at $a_1 = a_2 = \cdots = a_n$ We want to prove it for $n \in \mathbb{Z}^+$. Let's see a base case. For $n = 2$, we can start with

$$(\sqrt{x} - \sqrt{y}) \geq 0 \Rightarrow x + y \geq 2\sqrt{xy}$$

$$\frac{x + y}{2} \geq \sqrt{xy}$$

Now that we know our statement is true for $n = 2$, we can go on to assume our statement holds for some $k \in \mathbb{Z}^+$

$$\frac{a_1 + a_2 + \cdots + a_k}{k} \geq (a_1 a_2 \cdots a_k)^{\frac{1}{k}}$$

We are going to show that if our inductive hypothesis holds for some $k \in \mathbb{Z}^+$, then it also holds for $2k$ now :

$$\frac{a_1 + a_2 + \cdots + a_{2k}}{2k} = \frac{\frac{a_1+a_2+\cdots+a_k}{k} + \frac{a_{k+1}+a_{k+2}+\cdots+a_{2k}}{k}}{2}$$

Now, we can apply the inductive hypothesis on the first term in the numerator and the 2nd term in the numerator

$$\text{AM}_{2k} \geq \frac{\sqrt[k]{a_1 a_2 \cdots a_k} + \sqrt[k]{a_{k+1} a_{k+2} \cdots a_{2k}}}{2}$$

Now, we can apply AM-GM for 2 variables to get

$$\text{AM}_{2k} \geq \sqrt{\sqrt[k]{a_1 a_2 \cdots a_k} \times \sqrt[k]{a_{k+1} a_{k+2} \cdots a_{2k}}}$$

$$\text{AM}_{2k} \geq \sqrt[2k]{a_1 a_2 \cdots a_{2k}}$$

This proves that assuming $\text{AM}_k \geq \text{GM}_k$, we can prove that $\text{AM}_{2k} \geq \text{GM}_{2k}$. Now we'll "Fill the gaps". As we assumed that $\text{AM}_k \geq \text{GM}_k$ holds for any $a_k$ (by this I mean any $a_1$, $a_2, \cdots a_n$ etc), it would also hold if

$$a_k = \frac{a_1 + a_2 + \cdots a_{k-1}}{k-1}$$

Starting from our inductive hypothesis and replacing $a_k$

$$\frac{a_1 + a_2 + \cdots + \frac{a_1+\cdots+a_{k-1}}{k-1}}{k} \geq \sqrt[k]{a_1 a_2 \cdots a_{k-1} \left( \frac{a_1 + \cdots + a_{k-1}}{k-1} \right)}$$

Now, using a bit of algebra we can simplify and say

$$\frac{a_1 + a_2 + \cdots + \frac{a_1+\cdots+a_{k-1}}{k-1}}{k} = \frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1}$$

$$\left( \frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1} \right)^k \geq a_1 a_2 \cdots a_{k-1} \left( \frac{a_1 + \cdots + a_{k-1}}{k-1} \right)$$

$$\left( \frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1} \right)^{k-1} \geq a_1 a_2 \cdots a_{k-1}$$

If we take the $(k-1)$th root of both sides we get

$$\frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1} \geq \sqrt[k-1]{a_1 a_2 \cdots a_{k-1}}$$

Therefore we've showed, using $\mathrm{AM}_k \geq \mathrm{GM}_k$, that $\mathrm{AM}_{k-1} \geq \mathrm{GM}_{k-1}$. As we proved our base case, we have shown using Forward Backward induction that $\mathrm{AM}_k \geq \mathrm{GM}_k$ $\forall k \in \mathbb{Z}^+$. Therefore our speculation that we had from looking at an inequality that held for $n = 2, 3, 4$ led us to prove one of the most important inequalities in mathematics. Stay vigilant, there are theorems hiding in plain sight. Make observations, try some cases and seek generalizations.

**2nd Proof : (by George Polya)**

To make this article a bit interesting, let's see a proof that's not usually shown

$$\text{Recall } \; e^{x-1} \geq x \quad \forall x \in \mathbb{R}$$

with equality at $x = 1$. If you're not aware of this, don't worry we'll prove it now. Fun Fact, the inspiration for this inequality came to Polya in a dream as he stated.

$$f(x) = e^x - x \qquad f'(x) = e^x - 1 \qquad f''(x) = e^x$$

We see that $f'(x) = 0$ has one solution which means there's only one turning point (which occurs at $x = 0$) and at this turning point, $f''(x) > 0$. This means we have a global minimum at $x = 0$. $f(0) = 1$ so therefore $f(x) \geq 1$ $\forall x \in \mathbb{R}$.

$$e^x - x \geq 1 \Rightarrow e^x \geq 1 + x$$

If we consider the change of variables $x \to x - 1$, the equality case changes from $x = 0$ to $x = 1$ and we get

$$e^{x-1} \geq x \quad \forall x \in \mathbb{R}$$

This is quite an important inequality in general as well. Now we can start with our proof, let's consider $A$ and $G$ such that

$$A = p_1a_1 + p_2a_2 + \cdots p_na_n \qquad G = a_1^{p_1} \times a_2^{p_2} \cdots a_n^{p_n}$$

Where $a_i, p_i \in \mathbb{R}^+$. Let's plug $x = \frac{a_k}{A}$ into the inequality

$$e^{\left(\frac{a_k}{A} - 1\right)} \geq \frac{a_k}{A}$$

Raising both sides to the power $p_k$

$$e^{p_k\left(\frac{a_k}{A}-1\right)} \geq \left(\frac{a_k}{A}\right)^{p_k}$$

Recall that $p_k$ is positive so we don't need to worry about the inequality sign flipping

$$e^{p_k\left(\frac{a_k}{A}-1\right)} \geq \left(\frac{a_k}{A}\right)^{p_k}$$

Which holds for each $a_k$ and $p_k$. So let's multiply all of them together (the inequality for $k = 1, 2, 3, \cdots n$)

$$\exp\left(\sum_{k=1}^{n} \frac{p_k a_k}{A} - p_k\right) \geq \left(\frac{a_1}{A}\right)^{p_1} \left(\frac{a_2}{A}\right)^{p_2} \cdots \left(\frac{a_n}{A}\right)^{p_n}$$

$$\sum_{k=1}^{n} \frac{p_k a_k}{A} = \frac{p_1a_1 + p_2a_2 + \cdots p_na_n}{A} = 1$$

According to the definition of $A$, now we also have

$$-\sum_{k=1}^{n} p_k$$

in the exponent, so to simplify this, let's say it's 1. This means from now on we're assuming $p_1 + p_2 + \cdots p_n = 1$ According to the last slide

$$\exp\left(\sum_{k=1}^{n} \frac{p_k a_k}{A} - p_k\right) = \exp(0) = 1$$

So now we're left with

$$1 \geq \left(\frac{a_1}{A}\right)^{p_1} \left(\frac{a_2}{A}\right)^{p_2} \cdots \left(\frac{a_n}{A}\right)^{p_n}$$

Multiply both sides by $A^{p_1+p_2+\cdots+p_n}$ (which is just $A$)

$$A \geq a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}$$

$$p_1a_1 + p_2a_2 + \cdots p_na_n \geq a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n}$$

When $p_1 + p_2 + \cdots p_n = 1$. This is known as the **weighted** AM-GM inequality. If $p_1 = p_2 = \cdots p_n = \frac{1}{n}$

$$\frac{a_1 + a_2 \cdots a_n}{n} \geq \sqrt[n]{a_1a_2\cdots a_n}$$

Equality was at $x = 1 \Rightarrow \frac{a_k}{A} = 1 \Rightarrow a_1 = a_2 = \cdots = a_n$

Now we will prove **HM-GM** using **AM-GM**. Let $a_n = 1/x_n$

$$\frac{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}{n} \geq \sqrt[n]{\frac{1}{x_1}\cdot\frac{1}{x_2}\cdots\frac{1}{x_n}}$$

If we reciprocate both sides(this flips inequality sign), we get

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \leq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$$

This proves the **HM-GM** chain, the **AM-QM** chain will be proved later using the **Cauchy-Schwarz** inequality, which will be discussed right now

**Theorem Cauchy Schwarz Inequality : –**

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots a_nb_n)^2$$

For all real numbers $a_i$ and $b_i$ with equality at $a_i = \lambda b_i$ for all $i$

**Proof : (Using Dot products)**

To make this article even more interesting, let's show a proof that's not usually shown

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$$

where $\theta$ is the acute angle between the vectors **a** and **b** $\cos(\theta)$ can be negative sometimes, and in inequalities we try to avoid different signs so let's take absolute value on both sides, and let's notice that $|\cos(\theta)| \leq 1$. This means

$$|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}||\mathbf{b}||\cos(\theta)| \leq |\mathbf{a}||\mathbf{b}|$$

Now, writing this in component form

$$|\langle a_1, \cdots a_n\rangle \cdot \langle b_1, \cdots b_n\rangle| \leq |\langle a_1, \cdots a_n\rangle||\langle b_1, \cdots b_n\rangle|$$

We'll use the definition of the dot product next

$$\text{Recall that} \quad \langle a_1, \cdots a_n \rangle \cdot \langle b_1, \cdots b_n \rangle = \sum_{k=1}^{n} a_k b_k$$

And also that

$$|\langle a_1, \cdots a_n \rangle| = \sqrt{\sum_{k=1}^{n} a_k^2} \qquad |\langle b_1, \cdots b_n \rangle| = \sqrt{\sum_{k=1}^{n} b_k^2}$$

Using our inequality on the last slide, we get

$$\left| \sum_{k=1}^{n} a_k b_k \right| \leq \sqrt{\sum_{k=1}^{n} a_k^2} \times \sqrt{\sum_{k=1}^{n} b_k^2}$$

Squaring both sides yields our inequality,

$$\left( \sum_{k=1}^{n} a_k^2 \right) \left( \sum_{k=1}^{n} b_k^2 \right) \geq \left( \sum_{k=1}^{n} a_k b_k \right)^2$$

With equality at $|\cos(\theta)| = 1$ which happens when the vectors are parallel to each other, or $a_i = kb_i$ for all $i$. Let's put $b_i = 1$ for all $i$ and write $a_i$ as $x_i$ to get

$$(x_1^2 + x_2^2 + \cdots + x_n^2)n \geq (x_1 + x_2 + \cdots + x_n)^2$$

Dividing both sides by $n^2$ we get

$$\frac{(x_1^2 + x_2^2 + \cdots + x_n^2)}{n} \geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{n^2}$$

Square root-ing both sides gives us the **AM-QM** chain

$$\sqrt{\frac{(x_1^2 + x_2^2 + \cdots + x_n^2)}{n}} \geq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

That completes the **HM-GM-AM-QM** chain :)

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