Attempt any four questions. All questions carry equal marks.

1. Find the inverse of the element \( \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix} \) in \( GL(2, \mathbb{Z}_7) \).
   
   Let \( G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \mid a \in R, a \neq 0 \right\} \). Show that \( G \) is a group under matrix multiplication.

2. Prove that a group \( G \) is Abelian if and only if \((ab)^{-1} = a^{-1}b^{-1}\) for all \( a \) and \( b \) in \( G \).
   
   How many subgroups are there, of cyclic group \( \mathbb{Z}_{40} \). Find them and write the elements of all the subgroups.

3. Let \( \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix} \) and \( \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{pmatrix} \). Compute each of the following
   
   (i) \( \alpha^{-1} \),  (ii) \( \beta \alpha \),  (iii) \( \alpha \beta \).
   
   Also find the orders of \( \alpha, \beta, \alpha^{-1}, \beta \alpha, \alpha \beta \).

4. Show that the set \( S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbb{Z} \right\} \) is a subring of the ring \( M_2 \) of \( 2 \times 2 \) matrices over integers.
   
   Prove that the only ideals of a field \( F \) are \( \{0\} \) and \( F \) itself.

5. Let \( V = \{a_2x^2 + a_1x + a_0 : a_0, a_1, a_2 \in \mathbb{R} \} \) be a vector space over \( \mathbb{R} \). Show that \( \{x^2 + x + 1, x + 5, 3\} \) is a basis of \( V \).
   
   Let \( V = \mathbb{R}^3 \) and \( W = \{(a, b, c) : a, b, c \in \mathbb{R} \text{ and } a + b = c \} \). Is \( W \) a subspace of \( V \)? If so, what is its dimension?

6. Which of the following maps \( T: \mathbb{R}^2 \to \mathbb{R}^2 \) are linear transformations?
   
   (i) \( T(x_1, x_2) = (1 + x_1, x_2) \),  (ii) \( T(x_1, x_2) = (x_2 - x_1, 0) \).

   Let \( T: \mathbb{R}^3 \to \mathbb{R}^2 \) be a linear transformation such that \( T(2,1,0) = (1,2), T(1,0,−1) = (−1,1) \) and \( T(0,3,1) = (1,3) \). Find \( T(5,3,−1) \).