Name of the Course : **B.Sc. (Prog.) / Mathematical Sciences -(LOCF)**

Semester : **III**

Unique Paper Code : **42354302**

Name of the Paper : **Paper III-Algebra**

Duration: **3 Hours** Maximum Marks: **75**

Attempt any four questions. All questions carry equal marks. All symbols have usual meaning.

1. Prove that $\left\{\begin{bmatrix} x & y \\ z & w \end{bmatrix} : x, y, z, w \in \mathbb{Z}_{11}\right\}$ is a group under matrix multiplication. Is this group abelian? Give an example to verify your answer.

Find the inverse of $\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$ in $\mathrm{GL}(2, \mathbb{Z}_{11})$.

2. Find the order of the group and order of each element in the group where the groups are $\mathbb{Z}_{12}$ and $U(10)$.

Show that $U(14) = \langle 3 \rangle = \langle 5 \rangle$.

Show that $U(20) \neq \langle k \rangle$ for any $k \in U(20)$.

3. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{bmatrix}$. Compute A^{-1}, AB, BA.

Let $\alpha, \beta \in S_{n,}$ where S_n is a group of symmetries of degree n. Prove that
i) $\alpha^{-1} \beta^{-1} \alpha \beta$ is an even permutation.
ii) $\beta\alpha\beta^{-1}$ and α are both even or both odd.

4. Let $M_2(\mathbb{Z})$ be the ring of all 2 x 2 matrices over the integers and let

$$S = \left\{\begin{bmatrix} x & x-y \\ x-y & y \end{bmatrix} : x, y \in \mathbb{Z}\right\}.$$

Prove or disprove that S is a subring of $M_2(\mathbb{Z})$.

Consider the ring D as follows

$$D = \{a + ib + cj + dk \ : a, b, c, d \in \mathbb{R}\}$$

where $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji, jk = i = -kj, ik = j = -ki$. Prove or disprove that D forms a field.

Find a zero-divisor and a non-zero element a satisfying $a^2 = a$ (other than 1) in the ring $\mathbb{Z}_5[i]$, where $\mathbb{Z}_5[i] = \{x + iy : x, y \in \mathbb{Z}_5, i^2 = -1\}$.

5. Let W be the subset of $\mathbb{R}^4$, where

$$W = \{(a, b, c, d) \in \mathbb{R}^4 : c = a + 2b \text{ and } d = a - 3b\}.$$

i) Check whether W is a subspace of $\mathbb{R}^4$ or not.
ii) Find a basis of W and give its dimension.

Consider the set of vectors S from $\mathbb{R}^3$ such that

$$S = \{(2,2,3), (-1,-2,1), (0,1,0)\}.$$

Check whether S spans $\mathbb{R}^3$. Is S linearly independent?

6. Let $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be defined by $L(v_1, v_2) = (v_1 - v_2, v_1, 2v_1 + v_2)$.

i) Prove that L is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.

ii) Find basis for the Null space $N(L)$ of L and compute Nullity.

iii) Find Rank of L. Further, verify Rank Nullity Theorem.