1. Prove that \(\begin{bmatrix} x & y \\ z & w \end{bmatrix} : x, y, z, w \in \mathbb{Z} \) is a group under matrix multiplication. Is this group abelian? Give an example to verify your answer.

Find the inverse of \(\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}\) in \(GL(2, \mathbb{Z}_{11})\).

2. Find the order of the group and order of each element in the group where the groups are \(\mathbb{Z}_{12}\) and \(U(10)\).

Show that \(U(14) = \langle 3 \rangle = \langle 5 \rangle\).

Show that \(U(20) \neq \langle k \rangle\) for any \(k \in U(20)\).

3. Let \(A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix}\) and \(B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{bmatrix}\). Compute \(A^{-1}, AB, BA\).

Let \(\alpha, \beta \in S_n\), where \(S_n\) is a group of symmetries of degree \(n\). Prove that
i) \(\alpha^{-1} \beta^{-1} \alpha \beta\) is an even permutation.
ii) \(\beta \alpha \beta^{-1}\) and \(\alpha\) are both even or both odd.

4. Let \(M_2(\mathbb{Z})\) be the ring of all 2 x 2 matrices over the integers and let
\[ S = \left\{ \begin{bmatrix} x & y \\ x-y & y \end{bmatrix} : x, y \in \mathbb{Z} \right\}. \]

Prove or disprove that \(S\) is a subring of \(M_2(\mathbb{Z})\).

Consider the ring \(D\) as follows
\[ D = \{ a + ib + cj + dk : a, b, c, d \in \mathbb{R} \} \]
where \(i^2 = j^2 = k^2 = -1\) and \(ij = k = -ji, jk = i = -kj, ik = j = -ki\). Prove or disprove that \(D\) forms a field.

Find a zero-divisor and a non-zero element \(a\) satisfying \(a^2 = a\) (other than 1) in the ring \(\mathbb{Z}_5[i]\), where \(\mathbb{Z}_5[i] = \{ x + iy : x, y \in \mathbb{Z}_5, i^2 = -1 \}\).

5. Let \(W\) be the subset of \(\mathbb{R}^4\), where
\[ W = \{ (a, b, c, d) \in \mathbb{R}^4 : c = a + 2b and \, d = a - 3b \}. \]

i) Check whether \(W\) is a subspace of \(\mathbb{R}^4\) or not.

ii) Find a basis of \(W\) and give its dimension.
Consider the set of vectors $S$ from $\mathbb{R}^3$ such that
\[ S = \{(2, 2, 3), (-1, -2, 1), (0, 1, 0)\}. \]
Check whether $S$ spans $\mathbb{R}^3$. Is $S$ linearly independent?

6. Let $L : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined by $L(v_1, v_2) = (v_1 - v_2, v_1, 2v_1 + v_2)$.
   i) Prove that $L$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.
   ii) Find basis for the Null space $N(L)$ of $L$ and compute Nullity.
   iii) Find Rank of $L$. Further, verify Rank Nullity Theorem.