Name of Course	: **CBCS B.Sc. (H) Mathematics**
Unique Paper Code	: **32351302_OC**
Name of Paper	: **C6-Group Theory-I**
Semester	: **III**
Duration	: **3 hours**
Maximum Marks	: **75 Marks**

Attempt any four questions. All questions carry equal marks.

1. Let $G = \mathrm{GL}(n, \mathbb{R})$. Let $H = \{A \in G \mid \det A \text{ is a power of } 5\}$. Then prove or disprove that H is a subgroup of G. Find the elements in $U(10)$ and $U(12)$ that satisfy the equation $x^2 = 1$.

2. List all the elements of order 3 in $\mathbb{Z}_{24}$. Find the smallest subgroup of $\mathbb{Z}$ containing 12 and 18. Determine the subgroup lattice for $\mathbb{Z}_{24}$.

3. Let S_n be the symmetric group of degree n. Suppose that $\alpha \in S_n$ can be written as a product of disjoint cyclic permutations of lengths $m_1, m_2, \dots, m_r, (r \in \mathbb{N})$, respectively. Then prove that the order of α is $\mathrm{lcm}(m_1, m_2, \dots, m_r)$. Find the orders of (13)(27)(456)(8)(1237)(648)(5) and (124) (345). Furthermore, show that if H is a subgroup of S_n then either every member of H is an even permutation or exactly half of them are even. Also, find $Z(S_n)$ for $n \geq 3$.

4. Show that for a finite group G, the index of a subgroup H in G is $|G|/|H|$. Prove that every subgroup of index 2 of a group G is normal. Give an example of a subgroup H of index 3 in a group G which is not normal in G. Also, determine the index of $3\mathbb{Z}$ in $\mathbb{Z}$.

5. Let $H = \{\beta \in S_5 : \beta(1) = 1\}$ and $K = \{\beta \in S_5 : \beta(2) = 2\}$. Prove that H is isomorphic to K. Is the same true if S_5 is replaced by S_n, where $n \geq 3$? Further prove or disprove that S_4 is isomorphic to D_{12}.

6. If H is a subgroup of G and K is a normal subgroup of G, then prove that $H/(H \cap K)$ is isomorphic to HK/K. Also determine all homomorphisms from $\mathbb{Z}_n$ to itself.