Name of Course                  : CBCS B.Sc. Hons Mathematics
Unique Paper Code             : 32351302
Name of Paper                     : BMATH306-Group Theory-1
Semester                              : III
Duration                              : 3 hours
Maximum Marks                 : 75 marks

Attempt any four questions. All questions carry equal marks.

1. Let \( A \) be a non-empty set and \( \langle G, \cdot \rangle \) be a group. Let \( F \) be the set of all functions from \( A \) to \( G \). Define an operation \( * \) on \( F \) as follows:
   
   \[
   (f * g)(x) = f(x) \cdot g(x) \quad \forall x \in A.
   \]

   Prove that \( \langle F, * \rangle \) is a group.

Find the inverse of \[
\begin{bmatrix}
2 & 1 \\
4 & 3
\end{bmatrix}
\]
in \( GL(2, \mathbb{Z}_5) \), the group of \( 2 \times 2 \) non-singular matrices over \( \mathbb{Z}_5 \). Verify the answer by direct calculation.

Describe the group of symmetries of a non-square rectangle and draw its Cayley’s table.

2. Let \( a \) be an element of a group such that \( |a| = 3 \), prove that \( C(a) = C(a^2) \). Give an example to show that the conclusion fails if \( |a| = 4 \).

Find the orders of each of the elements of \( U(14) \). Show that it is cyclic and find all its generators.

3. Define Centre \( Z(G) \) of a group \( G \) and prove that \( Z(S_4) = \{e\} \).

For \( n > 2 \), show that every even permutation in \( S_n \) is a product of 3-cycles.

Let \( \sigma = (1,5,7)(2,5,3)(1,6) \). Express \( \sigma^{17} \) as a cycle.

4. Prove or disprove any six, stating the results used
   
   (i) \( \langle \mathbb{R}, + \rangle \approx \langle \mathbb{Q}, + \rangle \),
   (ii) \( \langle \mathbb{Q}, + \rangle \approx \langle \mathbb{Z}, + \rangle \),
   (iii) \( \langle \mathbb{R}, + \rangle \approx \langle \mathbb{R}^+, \cdot \rangle \),
   (iv) \( D_4 \approx \) Group \( Q \) of Quaternions,
   (v) \( U(20) \approx D_4 \),
   (vi) \( U(8) \approx U(12) \),
   (vii) \( U(10) \approx \mathbb{Z}_4 \),
   (viii) \( \frac{GL(2, \mathbb{R})}{SL(2, \mathbb{R})} \approx \mathbb{R}^+ \).

5. Let \( H \) be a subgroup of a group \( G \). Prove that \( aH \mapsto Ha^{-1} \) is a bijective mapping from the set of all left cosets of \( H \) in \( G \) to the set of all right cosets of \( H \) in \( G \). Can the same be said for \( aH \mapsto Ha? \)

   If \( G \) is a non-abelian group of order 8 with \( Z(G) \neq \{e\} \), prove that \( |Z(G)| = 2 \).

6. Let \( N \) be a normal subgroup of \( G \) and \( M \) be a normal subgroup of \( N \). If \( N \) is cyclic, prove that \( M \) is a normal subgroup of \( G \). Show by an example that the conclusion fails to hold if \( N \) is not cyclic.

   If \( \phi \) is a homomorphism from a finite group \( G \) to a finite group \( G' \), prove that \( |\phi(G)| \) divides the gcd of \( |G| \) and \( |G'| \).