Name of Course	: **CBCS B.Sc. (H) Mathematics**
Unique Paper Code	: **32351602**
Name of Paper	: **C14-Ring Theory and Linear Algebra-II**
Semester	: **VI**
Duration	: **3 hours**
Maximum Marks	: **75 Marks**

Attempt any four questions. All questions carry equal marks.

1. Prove that $\mathbb{Z}[x]$ is not a principal ideal domain. Also show that $2x^2 + x + 1$ is irreducible over $\mathbb{Z}_3$. Construct a field of order 16.

2. Prove that in a unique factorization domain, an element is irreducible if and only if it is prime. Prove or disprove that a subdomain of a principal ideal domain is a principal ideal domain. Show that $x^4 + 1$ is irreducible over $\mathbb{Q}$ but reducible over $\mathbb{R}$.

3. Let $T: P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ be a linear operator such that

$$T(f(x)) = f'(x) + f''(x).$$

Find the eigenvalues of T and their corresponding eigenspaces. Is T a diagonalizable linear operator? Find the minimal polynomial of T. Now suppose that $V = \mathbb{R}^3$ and$\beta = \{(1,0,2), (0,1,1), (1,1,0)\}$ be an ordered basis for V.Find an ordered basis $\beta^*$of V^* which is the dual basis corresponding to β.

4. Find an ordered basis for the T-cyclic subspace W of $\mathbb{R}^4$ generated by the vector z where $T: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is a linear operator such that

$$T(a, b, c, d) = (c + d, -b, a + b, 2a + b)$$

and $z = e_1$.Is W a T-invariant subspace of $\mathbb{R}^4$?Find the characteristic polynomial of T_W.Show that the characteristic polynomial of T_W obtained above divides the characteristic polynomial of T.Verify Cayley-Hamilton Theorem for T_W.

5. Apply the Gram-Schmidt process to the subset

$$S = \{f_1, f_2, f_3\}$$

of the inner product space $V = C[-\pi, \pi]$ with the inner product given by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)\, dt$$

to obtain an orthogonal basis forspan(S), i.e., the subspace of V spanned by the functions in S, where $f_1(x) = 1$, $f_2(x) = \sin x$ and $f_3(x) = \cos x$. Then normalize the vectors in this basis to obtain an orthonormal basis β for span(S).

6. Use the least squares approximation to find the best fit quadratic function for the set

$$\{(-1,5),(1,1),(2,1),(3,-3)\}.$$

Also compute the corresponding error E. Also find the minimal solution to the following system of linear equations:

$$x + y + z - w = 1$$

$$2x - y + w = 1$$