Name of Course : **CBCS B.Sc. (H) Mathematics**

Unique Paper Code : **32351403_OC**

Name of Paper : **C 10-Ring Theory and Linear Algebra-I**

Semester : **IV**

Duration : **3 hours**

Maximum Marks : **75 Marks**

Attempt any four questions. All questions carry equal marks.

1. For the following vectors in R^3, determine whether the first vector can be expressed as linear combination of the other two.

$$\{(5, 1, -5), (1, -2, -3), (-2, 3, -4)\}$$

Determine whether or not the set

$$\left\{\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$$

is linearly independent over Z_7.

Determine whether the set $\{1, (x-a), (x-a)^2, ..., (x-a)^n\}$ spans $P_n(F) = \{a_0 + a_1 x + ... + a_n x^n : a_0, a_1, ..., a_n \in F\}$, where a is fixed scalar.

2. Find three different bases of the subspace $W = \{(a_1, a_2, a_3) \in R^3 : a_1 - 4a_2 - a_3 = 0\}$ of R^3.

Let $W_1 = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R) : a + d = 0\right\}$ and $W_2 = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R) : d = 0, b + c = 0\right\}$. Find $\dim(W_1 \cap W_2)$ and $\dim(W_1 + W_2)$.

3. Let $\mathcal{M}_2(\mathbb{Z})$ denote the ring of all $2X2$ matrices with integer entries. Show that $\mathcal{M}_2(5\mathbb{Z})$ is an ideal of the ring $\mathcal{M}_2(\mathbb{Z})$.

Write all the elements of the quotient ring $\frac{\mathcal{M}_2(\mathbb{Z})}{\mathcal{M}_2(5\mathbb{Z})}$.

Find all idempotent elements, nilpotent elements, units and zero divisors of the ring $\mathbb{Z}_2 \oplus \mathbb{Z}_4$.

4. Let $\mathbb{Z}[x]$ denote the ring of polynomials with integer coefficients and let $I = \{f(x) \in \mathbb{Z}[x]\ such\ that\ f\ is\ an\ even\ integer\}$. Prove that $I = < x, 2 >$. Further prove that I is a prime ideal as well as a maximal ideal of the ring $\mathbb{Z}[x]$.

Show that $Q[\sqrt{2}]$ and $Q[\sqrt{3}]$ are not isomorphic rings.

5. If $T : R^2 \to R^2$ be a linear transformation for which $T(1,1) = (1,-2)$ and $T(-1,1) = (2,3)$ then what is $T(-1,5)$?

Let $T : R^3 \to R^3$ be a linear transformation defined by

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3),\ x_1, x_2, x_3 \in R.$$

Find a basis for Range space and and a basis for Null space of T and verify Dimension Theorem. Is T one-one? Is T onto? Is T invertible ? If so, find T^{-1}.

6. Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{pmatrix}$ be the matrix of linear operator T on R^3, w. r. t. standard ordered basis $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Now if $\gamma = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ be another basis of R^3, find a matrix B of T w. r. t. γ. Find a non-singular matrix Q such that $B = Q^{-1}AQ$.